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The Spectrum of a Ring as a Partially Ordered Set.

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Abstract

Let R be a commutative ring with identity, and let Spec R denote the set of prime ideals of R considered as a partially ordered set under inclusion. In his Commutative Rings, Kaplansky notes two properties of Spec R. The first (K1) is that in Spec R every chain has a least upper bound and a greatest lower bound. The second (K2) says that if $P \subset Q$ are distinct primes of R, then there exist distinct primes P_1 and Q_1 with $P \subset P_1 \subset Q_1 \subset Q$ such that there is no prime ideal properly between P_1 and Q_1. Kaplansky asks whether or not a partially ordered set S satisfying these two properties must be isomorphic as a partially ordered set to Spec R for some R.

This question is answered for two cases: In the first case S is assumed to be finite and in the second, S is a tree (i.e. if $s \in S$ then $\{x | x \leq s\}$ is a chain) with a unique minimal element.

Case (i) can be obtained from the recent Princeton thesis of M. Hochster who has characterized Spec R in terms of its topology. However, in contrast to Hochster's development, a direct, constructive proof for finite sets is given which yields some information about R itself.

It is known that if R is a Prüfer domain, then Spec R is a tree with unique minimal element and which, of course, satisfies properties (K1) and (K2). In case (ii) the converse statement is shown to be true, thus characterizing Spec R for Prüfer domains.

The relationship between this work and the work of Hochster is also investigated further.

Chapter I

Introduction

Let R be a commutative ring with identity and let Spec R denote the set of prime ideals of R considered as a partially ordered set under inclusion. (When we speak of a ring or domain, we shall always assume that it is commutative and that it has an identity.) In [6, page 6, Theorems 9 and 11], Kaplansky notes two properties of Spec R. The first (K1) is that in Spec R every totally ordered set of primes has a least upper bound and a greatest lower bound. The second (K2) says that if $P \subset Q$ are distinct primes of R then there exist distinct primes P_1 and Q_1 with $P \subset P_1 \subset Q_1 \subset Q$ such that there is no prime ideal properly between P_1 and Q_1. Kaplansky asks whether or not a partially ordered set S satisfying these two properties must be isomorphic as a partially ordered set to Spec R for some ring R. The object of this paper is to investigate this question.

The question is answered for two cases. In the first case, S is assumed to be finite and in the second, S is a tree (i.e. if $s \in S$ then $\{x \mid x \leq s\}$ is totally ordered) with a finite number of minimal elements. Finally a third method of approaching the problem is discussed.

In Chapter II, we show that any finite partially ordered set is isomorphic as a partially ordered set to the spectrum of some ring.

Our proof provides a way of building a ring with a desired spectrum. In establishing the finite case, we made extensive use of two theorems which, roughly speaking, are as follows:

(i) Let R be a domain containing a field and let R have a finite number of maximal ideals. If a partially ordered set X is the result of tying together the maximals of Spec R in some pattern, then there is a domain $S \subset R$ such that Spec $S \cong X$.

(ii) Let R be a domain containing a field k, and let R have n maximal ideals. If $D_1, \cdots, D_n$ are domains with quotient field k, then there is a domain $S \subset R$ such that Spec S is the result of attaching the minimal element of each Spec D_i to one of the maximal elements of Spec R.

If $f:X \to Y$ is an order preserving map between finite partially ordered sets, we also discuss the possibility of constructing Spec $R \cong X$ and Spec $S \cong Y$ in such a way that there is a homomorphism $f_*:S \to R$ which induces f.

In [4], M. Hochster studies the functor Spec from the category of commutative rings with an identity to the category of topological spaces and continuous functions. In his paper Hochster characterizes the spaces in the image of Spec as those topological spaces which are T_0; quasi-compact; the quasi-compact open subsets are closed under finite intersection and form an open basis; and every non-empty irreducible closed subset has a generic point.

If X is a finite partially ordered set, we can topologize X by letting a subbasis for the closed sets of the topology be the closures of points where for $x \in X$, $\overline{\{x\}} = \{y \in X \mid x \le y\}$. With this topology, X has all the topological properties listed above, so our result about finite sets can be obtained from Hochster's work. Our proof, however, is a simpler approach to the case we have answered and provides a method of constructing domains with a desired finite spectrum.

A domain R is a <u>Prüfer</u> domain if for every prime ideal P, R_P is a valuation ring. R is <u>Bezout</u> if every finitely generated ideal is principal. It is well known (see for example [6, page 38-39]) that a Bezout domain must be Prüfer. In the third chapter we characterize the possible Spectra of Prüfer and Bezout domains. If R is a Prüfer domain, Spec R must have the following property: (*) If $P_1 \subset P$ and $P_2 \subset P$ then either $P_1 \subset P_2$ or $P_2 \subset P_1$. We show that if X is a partially ordered set with a unique minimal element, for which properties (K1), (K2) and (*) hold, then there is a Bezout domain R such that Spec $R \cong X$. Our proof uses a theorem of Jaffard [5, page 78, Theorem 3] which says that every lattice ordered group is a group of divisibility. (We will define these terms later.)

In Chapter IV, we take note of another property of Spec R: (D) Every lower directed set has a greatest lower bound and every upper directed set has a least upper bound. This property can be obtained as a part of Hochster's work, [4], but a direct proof is quite simple and we give it. We do not know whether or not this is, in fact, a stronger property than (K1). We are able to show that if a partially ordered set X is countable or lattice (i.e., the inf and sup of any two elements exist) then (K1) and (D) are equivalent.

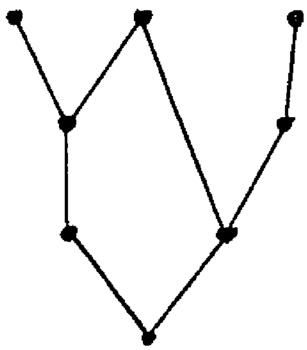
In general our notation will be that of [11]. One exception is that the ring R is not included in the "ideals" of R.

Chapter II

Finite Partially Ordered Sets

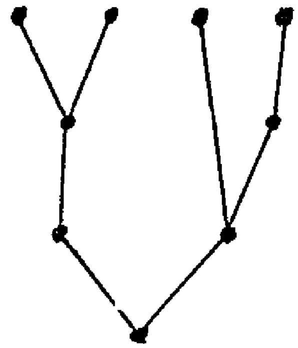
If X and Y are partially ordered sets, we say they are isomorphic if there is an order preserving bijection $f: X \to Y$ such that f^{-1} is also order preserving. We will show that if X is a finite partially ordered set, there is a ring R such that Spec $R \cong X$. We prove this first with the additional assumption that X has a unique minimal element. This part of the proof proceeds by induction on <u>dim X</u> $= \sup\{n \mid$ there is a chain $x_0 < x_1 < \cdots < x_n;\ x_i \in X\}$. If $x \in X$ we define <u>ht(x)</u>$=\sup\{n \mid$ there is a chain $x_0 < x_1 < \cdots < x_n = x;\ x_i \in X\}$.

The proof involves two main ideas. One is embodied in Theorem (2.6) and the other in Theorem (2.7). We can illustrate the role they play with the 3 dimensional partially ordered set X in Figure 1. We first "untie" the ordering at that maximal element which has two elements immediately below it, and obtain Y as in Figure 2. If we let $Z = \{y \in Y \mid ht(y) \le 2\}$ we get Figure 3 and dim $Z = 2$. At this point in our proof we use the induction hypothesis to construct a ring R_0 such that Spec $R_0 \cong Z$. If we equip R_0 with sufficient additional properties, Theorem (2.7) provides a ring $R_1 \subset R_0$ such that Spec $R_1 \cong Y$ and Theorem (2.6) provides a ring $R_2 \subset R_1$ such that Spec $R_2 \cong X$.



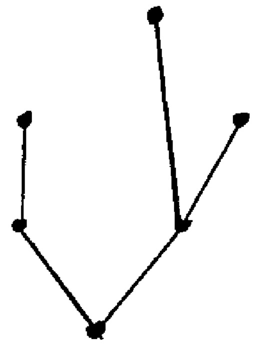
X

Figure 1



Y

Figure 2



Z

Figure 3

In order to be able to relate the spectrum of one ring to that of another, we have made use of the concept of explicitness. If $S \subset R$, we say that <u>R is explicit over S</u> if $R = S_{(u(R)\cap S)}$ where $u(R)$ is the units of R. If $R = \bigcap_{i=1}^{n} R_i$, <u>the intersection is explicit</u> if each R_i is explicit over R. The usual terminology, as in [11] or [1], is that R_2 is a quotient ring of R_1 with respect to a multiplicative system. The word explicit, however, seems to be a useful abbreviation, and it has a historical precedent in the works of Krull, for example [8, page 559].

Certain facts about explicitness are easy to see. If R is explicit over S and S is explict over T then R is explicit over T. If $T \subset S \subset R$ and R is explicit over T then R is explicit over S. If $T \subset S \subset R$ and R is explicit over T, it does not follow that S is explicit over T. To see this let R be a field and choose domains $T \subset S$ with quotient field T such that S is not a quotient ring of T. The following theorem gives some information about explicit intersections.

(2.1) <u>Theorem</u>. Let $R_1, \cdots, R_n$ be domains, each having a finite number of maximal ideals. Let $R = \bigcap_{i=1}^{n} R_i$ and let T be a domain contained in R. If R_i is explicit over T for all i then R is explicit over T.

<u>Proof</u>. If we localize at each maximal ideal of each R_i, we get a collection of quasi-local domains, each explicit over T. Their intersection is R. We can reduce this set to one which has no containment relationships and whose intersection is still R. It will suffice to prove the theorem for this case. Thus we assume that for all i, R_i is quasi-local and $R_i \not\subset R_j$ if $i \neq j$. Let M_i be the maximal ideal of R_i and let $M_i \cap T = P_i$. Since R_i is explicit over T, we have $T_{P_i} = R_i$ for all i and thus if $i \neq j$ then $P_i \not\subset P_j$.

Let $x \in R$. We need to find a $u \in u(R) \cap T$ such that $ux \in T$. For each i there is a $v_i \in u(R_i) \cap T$ such that $v_i x \in T$ since each R_i is explicit over R. For each i we can also choose a $w_i \in \bigcap_{j \neq i} P_j \backslash P_i$. Let $u_i = w_i v_i$. Then $u_i \in u(R_i) \cap T \cap (\bigcap_{j \neq i} P_j)$ and $u_i x = w_i v_i x \in T$. If $u = \sum_{i=1}^{n} u_i$ then $u \in T$, $ux \in T$ and for each i, $u = u_i + \sum_{j \neq i} u_j$ where $u_i \in u(R_i)$. Now $\sum_{j \neq i} u_j \in P_i \subset M_i$, thus $u \in u(R_i)$ for all i and $u \in u(R)$. Thus R is explicit over T and the proof is finished.

Let $R_1, \cdots, R_n$ and $S_1, \cdots, S_n$ be domains such that $S_i \subset R_i$ for all i. We wish to establish conditions under which $R = \bigcap_{i=1}^{n} R_i$ being explicit would imply $S = \bigcap_{i=1}^{n} S_i$ is explicit. For any domain R, we shall use J(R) to represent the intersection of the maximal ideals of R, i.e. <u>the Jacobson radical of R</u>. We prove first the following lemma.

(2.2) <u>Lemma</u>. Let R_1, R, S_1, S be domains such that S is contained in $S_1 \cap R$ and S_1 and R are contained in R_1. Let R_1 have a finite number of maximal ideals and suppose that M maximal in R_1 implies $M \cap R$ maximal. Let

$J = J(R_1) \cap R$ and let A be an ideal of R such that $J + A = R$. If $J(R_1) \subset J(S_1)$ and $A \cap S_1 \subset S$, then R_1 explicit over R implies S_1 is explicit over S and $(A \cap S_1) + (J(R_1) \cap S) = S$.

<u>Proof</u>. Figure 4 represents the containment relationships existing between our domains. Given $x \in S_1$ we must find $u \in u(S_1) \cap S$ such that $ux \in S$. Since R_1 is explicit over R, there is a $u_1 \in u(R_1) \cap R$ such that $u_1 x \in R$. Choose $a \in A$ such that $(1-a) \in J$. This implies $a \in u(R_1)$, hence $au_1 \in u(R_1) \cap R$,

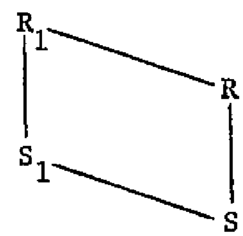


Figure 4

$au_1 \in A$, and $au_1 x \in A$. Since $J(R_1) \subset J(S_1)$, $(1-a) \in S_1$, hence $a \in S_1$ which implies $a \in A \cap S_1 \subset S$ and $(1-a) \in S \cap J(R_1)$. We conclude $(A \cap S_1) + (J(R_1) \cap S) = S$.

Let N be a maximal ideal of R which contains J. Since J is the intersection of the contractions to R of the maximal ideals of R_1, and since M maximal in R_1 implies $M \cap R$ is maximal, we conclude that N contains, hence equals, the contraction of some maximal ideal M in R_1. Now $au_1 \in u(R_1)$ thus there is no maximal ideal in R which contains au_1 and N. Thus $J + (au_1) = R$. Choose $v, u \in R$ such that $v \in J$, $u \in (au_1)$, and $v + u = 1$. Since $v \in J \subset J(S_1)$, we have $u \in u(S_1) \cap A \subset S$. Thus $u \in u(S_1) \cap S$ and $ux \in A \cap S_1 \subset S$. This concludes the proof.

(2.3) <u>Theorem</u>. Let $R_1, \cdots, R_n, S_1, \cdots, S_n$ be a collection of domains such that $R = \cap R_i$ and $S = \cap S_i$ and for all i, $S_i \subset R_i$. Suppose $J(R_i) \subset J(S_i)$ for all i, $(J(R_i) \cap R)$ for $i=1, \cdots, n$ are pairwise comaximal, and each R_i has a finite number of maximal ideals. For each j, if R_j is explicit over

R then S_j is explicit over S; moreover $J(R_j) \cap S$ and $J(R_i) \cap S$ are comaximal for any $i \neq j$.

<u>Proof</u>. If x is a non-unit of R, then it must be a non-unit of some R_i. Thus if M is a maximal ideal of R it is contained in the union of all the prime ideals which are contractions of the maximal ideals of the R_i's. By [11, page 215], M must be contained in one of them. Since M is maximal we may say that M is the contraction of a maximal ideal in some R_i. Since $(J(R_i) \cap R)$ for $i=1, \cdots, n$ are comaximal it is not possible for the contraction of a maximal in R_i to be contained in the contraction of a maximal in R_j if $i \neq j$. Thus if we have a j such that R_j is explicit over R, the maximals of R_j must contract to maximals of R. Let $A = \bigcap_{i \neq j} (J(R_i) \cap R)$ and we have satisfied the hypothesis of Lemma (2.2). We conclude S_j is explicit over S and $(J(R_j) \cap S) + (A \cap S_j) = S$. Since $(A \cap S_j) = (\bigcap_{i \neq j} (J(R_i) \cap R)) \cap S_j \subset (J(R_i) \cap R) \cap S$ for any $i \neq j$, we have $(J(R_j) \cap S) + (J(R_i) \cap S) = S$ for any $i \neq j$. This completes the proof.

(2.4) <u>Discussion</u>. Let X be a partially ordered set and ρ an equivalence relation on X for which $x\rho y$ and x non-maximal imply $x = y$. Let $Y = X/\rho$. We define an order on Y as follows: If y_1 and y_2 are in Y, then $y_1 \leq y_2$ if and only if there is a $x_1 \in y_1$ and $x_2 \in y_2$ such that $x_1 \leq x_2$. The "if" part of this statement is necessary if the canonical map $\Phi: X \to Y$ is to be order preserving. Thus the ordering on Y is the minimal one for which Φ is order preserving. We note two facts about Φ:

(i) $\Phi(x)$ is maximal if and only if x is maximal.

(ii) Φ restricted to the non-maximals of X is an isomorphism onto the non-maximals of Y.

We will say that Y is obtained by "tying together maximal elements of X." The basic idea is given in Figure 5. Here points represent elements, the line segments describe the ordering, and the m's are the maximal elements.

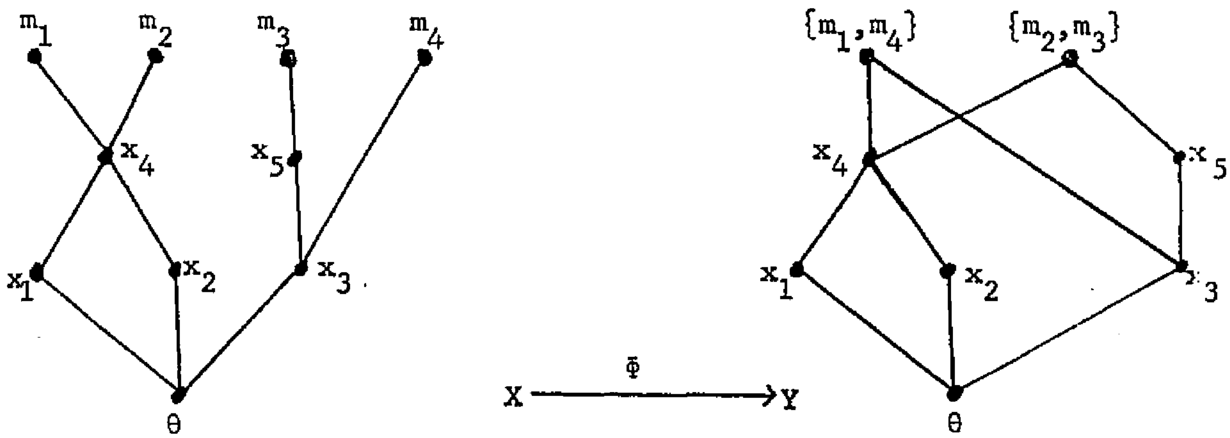


Figure 5

Now let X, Y be partially ordered sets and $\Phi: X \to Y$ an order preserving epimorphism satisfying (i) and (ii). If y_1 and $y_2 \in Y$ and $y_1 \leq y_2$ imply that there exist $x_1 \in \Phi^{-1}(y_1)$, $x_2 \in \Phi^{-1}(y_2)$ such that $x_1 \leq x_2$, then we say that "Φ is a tying together of maximal elements of X." We do this because there is an equivalence relation ρ defined on X by "$x \rho y$ if and only if $\Phi(x) = \Phi(y)$" which by (i) and (ii) has the property that $x \rho y$ and x non-maximal imply $x = y$. Further, Φ order preserving tells us that if $y_1, y_2 \in Y$, then $y_1 \leq y_2$ if and only if there is a $x_1 \in \Phi^{-1}(y_1)$ and $x_2 \in \Phi^{-1}(y_2)$ such that $x_1 \leq x_2$.

(2.5) <u>Theorem</u>. Let R be a domain with a finite number of maximal ideals. Let k be a field contained in R. If $S = k + J(R)$, then S is quasi-local and Spec S is the result of tying all the maximal ideals of Spec R together.

<u>Proof</u>. Let $J = J(R)$. Since $S/J \cong k$, J is a maximal ideal of S. If $r + j \in S$ for $r \neq 0 \in k$ and $j \in J$, then $r + j$ is a unit in R and $\frac{1}{(r+j)} = \frac{1}{r} - \frac{j}{r(r+j)}$. Now $\frac{1}{r} \in k$ and $\frac{1}{r(r+j)} \in u(R)$, thus $\frac{j}{r(r+j)}$ is in J and $\frac{1}{(r+j)} \in S$. Hence, S is quasi-local with maximal ideal J. Now let Φ:Spec $R \to$ Spec S be defined by intersection. Clearly Φ is order preserving. For any maximal ideal M in R, $\Phi(M) = J$. If $M_1, \cdots, M_n$ are the maximals of R and P is a prime ideal of R such that $P \cap S = J$, then $\bigcap_{i=1}^{n} M_i = J \subset P$. Thus $P = M_i$ for some i and $\Phi(P)$ is maximal if and only if P is maximal in R.

If $P \neq J$ is a prime ideal in S, choose $x \in J \setminus P$. We have $xR \subset J \subset S$, thus $R \subset S_P$ and $PS_P \cap R = Q$ is a prime ideal in R for which $Q \cap S = P$. Thus Φ is onto. Suppose Q' is a prime ideal of R and $Q' \cap S = P$. Since $R \subset S_P \subset R_{Q'}$, $Q'S_P$ lies over P, and hence $Q'S_P = PS_P$. Therefore $Q' = PS_P \cap R = Q$ and hence is uniquely determined by P. We have seen, therefore, Φ is an order preserving bijection of the non-maximals of Spec R to the non-maximals of Spec S. To finish the proof we must show that if $P_1 \subset P_2$ are prime ideals of S then there exist $Q_1 \in \Phi^{-1}(P_1), Q_2 \in \Phi^{-1}(P_2)$ such that $Q_1 \subset Q_2$. If $P_2 = J$, then this is trivial so assume $P_2 \neq J$. This implies $R \subset S_{P_2}$ and, of course, we choose $Q_1 = P_1S_{P_2} \cap R$ and $Q_2 = P_2S_{P_2} \cap R$. This completes the proof.

A very specialized case of the preceding appears in [7, page 52]. We see in the next theorem that this tying together of maximal ideals can be done in a more general way. A specialized version of it can be found in [10].

We shall need the following lemma in our proof of Theorem (2.6). A more general version of this lemma is [2, page 98, Proposition 1.2].

Lemma. Let $D_1, \cdots, D_n$ be domains with a finite number of maximal ideals such that each D_i is explicit over D. If $u(D) = \bigcap_{i=1}^{n} (u(D_i) \cap D)$, then $D = \bigcap_{i=1}^{n} D_i$.

Proof. By Theorem (2.1), $\bigcap_{i=1}^{n} D_i$ is explicit over D. Let $x \in u(\bigcap_{i=1}^{n} D_i) \cap D$; then $x \in u(D_i)$ for all i. Thus $x \in \bigcap_{i=1}^{n}(u(D_i) \cap D) = u(D)$ and $D = \bigcap_{i=1}^{n} D_i$.

(2.6) Theorem. Let R be a domain with a finite number of maximal ideals. Let k be a field contained in R. Let X be a partially ordered set and Φ:Spec R$\to$X a tying together of maximal elements of Spec R. Then there is a domain $S \subset R$ such that Spec $S \cong X$.

Proof. Let $m_1, \cdots, m_n$ be the maximal elements of X and let $s_i = R\backslash\{P \in$ Spec $R|\Phi(P) = m_i\}$ for $i = 1, \cdots, n$. Let $R_i = R_{s_i}$; then by the preceding lemma, $R = \cap R_i$ is an explicit intersection.

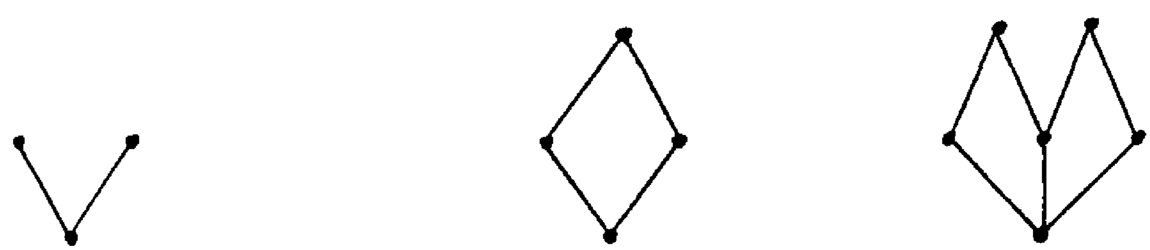
Now let $S_i = k + J(R_i)$ and $S = \bigcap_{i=1}^{n} S_i$. By Theorem (2.3), this is an explicit intersection and $(J(R_i) \cap S)$ for $i = 1, \cdots, n$ are pairwise comaximal in S. Since their union contains all non-units of S, they are precisely the maximal ideals of S. (We made a similar argument in Theorem (2.3)).

If we let $T = k + J(R)$, $T \subset S \subset R$ and we get maps f:Spec R$\to$Spec S and g:Spec S$\to$Spec T defined by intersection. Now $J(R_i) \cap S \subset J(R_i) \cap R = \cap\{P \in$ Spec $R|\Phi(P) = m_i\}$ tells us f maps the maximals of R onto the maximals of S such that if P is maximal in R, $f(P) = J(R_i) \cap S$ if and only if $\Phi(P) = m_i$. By Theorem (2.5), g and g(f) are onto and 1-1 on non-maximal ideals. As a result, f(P) is maximal if and only if P is maximal and f is an order preserving bijection of the non-maximals of Spec R to the non-maximals of

Spec S.

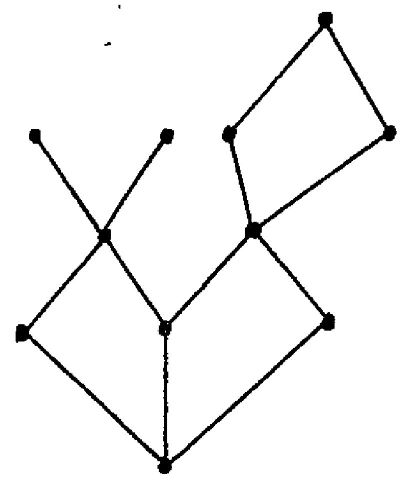
We now need to see that if $P_1 \subset P_2$ are primes of S then there are primes $Q_1 \in f^{-1}(P_1)$ and $Q_2 \in f^{-1}(P_2)$ such that $Q_1 \subset Q_2$. If P_2 is non-maximal, there are unique primes Q_1, Q_2 in R such that $f(Q_1) = P_1$ and $f(Q_2) = P_2$. Since $g(f(Q_1)) \subset g(f(Q_2))$ we must have $Q_1 \subset Q_2$. So let us assume $P_2 = J(R_i) \cap S$ for some i. If $P_1 = P_2$ it is trivial so we assume $P_1 \neq P_2$; hence $P_1S_i < J(R_i) \subset S_i \subset R_i$. As we have seen in a similar situation before, $R_i \subset {S_i}_{(P_1S_i)} = S_{P_1}$. Hence $P_1S_{P_1} \cap R \subset \cup\{P \in \text{Spec } R \mid \Phi(P) = m_i\}$ and thus contained in one of them. Set $Q_1 = P_1S_{P_1} \cap R$ and Q_2 equal to any maximal ideal containing Q_1 chosen from the set above. Clearly $f(Q_1) = P_1$ and $f(Q_2) = P_2$ and the proof is finished.

Now let us assume that we have domains D_1 and D_2 with quotient field k contained in the domain R, and that Figure 6 describes Spec D_1, Spec D_2, and Spec R. We will see in the next theorem that there is a domain $S \subset R$ such that Spec S, as in Figure 7, is the result of attaching the minimal elements of Spec D_1 and Spec D_2 to the maximal elements of Spec R.



Spec D_1 Spec D_2 Spec R

Figure 6



Spec S

Figure 7

We need the following lemma which is a part of Theorem 5.1 in [3, pages 247-249].

<u>Lemma</u>. Let D_o be a domain whose quotient field is contained in D where D is a quasi-local domain with maximal ideal M. Let $R = D_o + M$. If P is a prime ideal of R, then either $M \subset P$ or $P \subset M$.

<u>Proof</u>. Suppose $P \not\subset M$. Let x be chosen in $P \backslash M$. If $x = d + m$ for $d \neq 0 \in D_o$ and $m \in M$, then $\frac{1}{(d+m)}$ is a unit of R so for any $y \in M$, $\frac{y}{(d+m)} \in M \subset R$. Thus $M \subset xR \subset P$.

(2.7) <u>Theorem</u>. Let R be a domain with exactly n maximal ideals $M_1, \cdots,$ M_n. Let $D_1, \cdots, D_n$ be domains (not necessarily distinct) with quotient field $k \subset R$. Let $S_i = D_i + M_i R_{M_i}$ and $S = \bigcap_{i=1}^{n} S_i$. Let $U_i = \{P \in \text{Spec } S \mid P \supset (M_i \cap S)\}$, let $U = \bigcup_{i=1}^{n} U_i$, and let $L = \{P \in \text{Spec } S \mid P \subset (M_i \cap S) \text{ for some } i\}$.

Then,

(i) $U \cap L = \{M_i \cap S \mid i=1, \cdots, n\}$ and the $M_i \cap S$ for $i=1, \cdots, n$ are pairwise comaximal ideals of S.

(ii) $L \cong \text{Spec } R$.

(iii) $U_i \cong \text{Spec } D_i$ and if $i \neq j$, $U_i \cap U_j = \emptyset$.

(iv) If $P \in L$ and P is contained in some element of U_i then $P \subset M_i \cap S$.

(v) $U \cup L = \text{Spec } S$.

<u>Proof</u>. Let $k + MR_{M_i} = R_i$ and let $R' = \bigcap_{i=1}^{n} R_i$. If we review Theorem (2.6) and its proof, we see that $\text{Spec } R' \cong \text{Spec } R$. Now $S_i \subset R_i \subset R$ (see Figure 8) and, of course, $R = \bigcap_{i=1}^{n} R_{M_i}$, so by Theorem (2.3), $R' = \bigcap_{i=1}^{n} R_i$ and $S = \bigcap_{i=1}^{n} S_i$ are explicit intersections. Moreover, $(M_i R_{M_i} \cap S) = (M_i \cap S)$ for $i=1, \cdots, n$ are pairwise comaximal ideals of S.

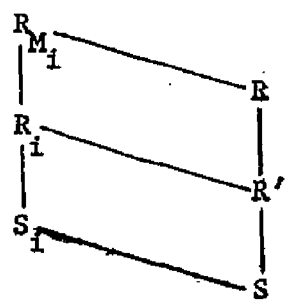


Figure 8

It follows that $U \cap L = \{M_i \cap S \mid i=1, \cdots, n\}$ and if $i \neq j$, $U_i \cap U_j = \emptyset$. Now since k is the quotient field of each D_i, R_i is explicit over S_i for all i and thus R_i is explicit over S for all i. By Theorem (2.1), R' is explicit over S. Since $u(R') \cap S = S \setminus \bigcup_{i=1}^{n} (M_i \cap S)$, $L \cong \text{Spec } R' \cong \text{Spec } R$.

Now if $x \in u(S_i) \cap S$ and $y \in (M_i \cap S)$, then $x + y \in u(S_i) \cap S$. Thus if $f: S \rightarrow S/(M_i \cap S)$ is the canonical map, then $f(u(S_i) \cap S) \subset u(S/(M_i \cap S))$. By the commutativity of the formation of residue class rings and quotient rings, we have $D_i \cong S_i/M_i R_{M_i} \cong S/(M_i \cap R)$. We conclude that $\text{Spec } D_i \cong$

Spec $(S/M_i \cap R) \cong U_i$.

For (iv) let $P \in L$ and let P be contained in some element of U_i. Then $P \cap (u(S_i) \cap S) = \emptyset$; hence PS_i is a prime ideal of S_i which lies over P. By the preceding lemma either $PS_i \subset M_iR_{M_i}$ or $M_iR_{M_i} \subset PS_i$. By (i) we must have $PS_i \subset M_iR_{M_i}$ and $P \subset (M_i \cap S)$.

Now let P be a prime ideal of S. We assert that there exist an i such that $u(S_i) \cap P = \emptyset$. Suppose not, then let $v_i \in u(S_i) \cap P$ be chosen for each i. Since the $(M_i \cap S)$'s are pairwise comaximal in S and $(M_i \cap S) \subset J(S_i) \cap S$ for each i, we have $J(S_i) \cap S$ and $\bigcap_{j\neq i} (M_j \cap S)$ are comaximal (see for example [11, page 177, Theorem 31]) for each i. Thus for each i we may choose w_i such that $w_i \in \bigcap_{j\neq i} (M_j \cap S) \cap (u(S_i) \cap S)$. If $u_i = v_iw_i$, then $u_i \in P \cap u(S_i) \cap (\bigcap_{j\neq i} (M_j \cap S))$. Let $u = \sum_{i=1}^{n} u_i$. Then $\sum_{j\neq i} u_j \in J(S_i)$ and $u_i \in u(S_i)$. Thus $u \in u(S_i)$ for all i and $u \in u(S) \cap P$. But this is impossible, so there exists an i for which $u(S_i) \cap P = \emptyset$. Thus PS_i is a prime ideal of S_i which contracts to P and applying our lemma again and contracting to S we get $P \subset M_i \cap S$ or $M_i \cap S \subset P$. Thus $U \cup L =$ Spec S. This completes the proof.

(2.8) <u>Discussion</u>. Let X be a finite partially ordered set. We define a related set $O(X)$ as follows: $O(X) = A \cup B$ where $A = \{x \in X | x \text{ is not maximal}\}$; $B = \{(x,m) | m \text{ is maximal, } x < m \text{ and there does not exist an element of } X \text{ properly between } x \text{ and } m\}$.

We can define an order on $O(X)$ in terms of the order on X as follows: (we use $\leq_o$ for the order on $O(X)$)

(i) If $x_1, x_2 \in A$, then $x_1 \leq_o x_2$ if and only if $x_1 \leq x_2$;

(ii) $(x_1,m_1) \leq_o (x_2,m_2)$ if and only if $x_1 = x_2$ and $m_1 = m_2$;

(iii) if $x \in A$, $(x_1,m_1) \in B$, then $x \leq_o (x_1,m_1)$ if and only if $x \leq x_1$.

It is clear that B is the set of all maximal elements of O(X) and if $x \in A$, $x < (x_1, m_1)$ and there is no element of O(X) properly between x and (x_1, m_1), then $x = x_1$. Define $\Phi: O(X) \to X$ by $\Phi(x) = x$ if $x \in A$ and $\Phi(x,m) = m$ if $(x,m) \in B$. It is immediate that Φ is a tying together of maximal elements of O(X). Figure 9 illustrates a possible X and the resulting O(X). We say that O(X) is obtained by "opening" or "untying" the maximal elements of X.

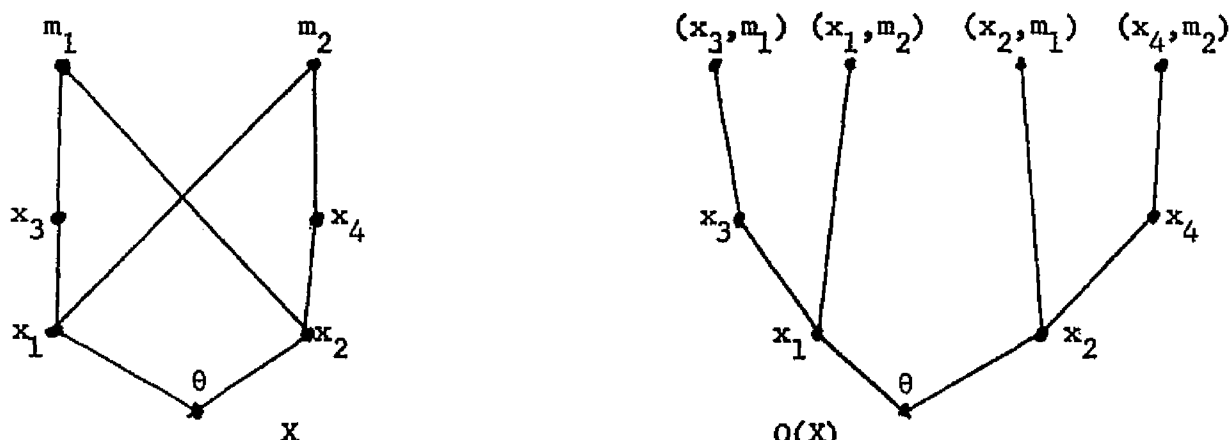


Figure 9

(2.9) <u>Theorem</u>. Let X be a finite partially ordered set with a unique minimal element. Then there exists a domain R such that Spec $R \cong X$.

<u>Proof</u>. The proof will proceed by induction on $\dim X = \sup\{n \mid \text{there is a chain } x_0 < x_1 < \cdots < x_n;\ x_i \in X\}$. Recall that $ht(x) = \sup\{x \mid \text{there is a chain } x_0 < x_1 < \cdots < x_n = x;\ x_i \in X\}$ for any $x \in X$.

For any $n = 0$, any field will do, so we will start with $n = 1$. Let k be any field, and suppose X has $m + 1$ elements including the minimal element.

Let $t_1, \cdots, t_m$ be indeterminants over k. Then we can obtain m independent, rank 1, discrete valuation rings $V_1, \cdots, V_m$ with quotient field $k(t_1, \cdots, t_m)$ such that $k \subset D = \bigcap_{i=1}^{m} V_i$. It is well known (for example [1, page 263, Theorem 18.8]) that D is a Prüfer domain and Spec $D \cong X$. Now let us assume that if $1 \leq n < r$, if $\dim X = n$, and if k is any field, then there exists a domain D such that $k \subset D$ and Spec $D \cong X$. Let $\dim X = r$, and let k be a fixed field. Let $Y = O(X)$ as defined in the previous section. Then $\dim Y = r$ and if y is a maximal element of Y then there is a unique $y_1 < y$ such that $y_2 < y$ implies $y_2 \leq y_1$. Let $Z = \{y \in Y | ht(y) \leq r - 1\}$. Then $\dim Z = r - 1$. Let $z_1, \cdots, z_m$ be the maximal elements of Z. If $y \in Y \backslash Z$, then there is a unique $z_i < y$ such that if $y' \in Y$, with $y' < y$ then $y' \leq z_i$. For each $i = 1, \cdots, m$, let $U_i = \{y \in Y | y \geq z_i\}$. Either $U_i = \{z_i\}$ or U_i is a one dimensional partially ordered set. Using the case of $n = 1$, we choose a field $K \supset k$ and domains $D_1, \cdots, D_m$ with quotient field K such that $k \subset \bigcap_{i=1}^{m} D_i$ and for each i, Spec $D_i \cong U_i$. (If U_i has only one point z_i, then $D_i = K$.) Now by the induction hypothesis there is a domain R such that $K \subset R$ and Spec $R \cong Z$. By Theorem (2.7) there is a domain $S \subset R$ such that Spec $S \cong Y$ and since $k \subset \bigcap_{i=1}^{m} D_i$ we have $k \subset S$. By Theorem (2.6) there is a domain $T \subset S$ such that Spec $T \cong X$. We conclude that every finite partially ordered set with a unique minimal element is isomorphic to the spectrum of some domain.

(2.10) Theorem. Let X be a finite partially ordered set. Then there exists a ring R such that Spec $R \cong X$.

Proof. Let $Y = X \cup \{\theta\}$ ordered so that $\theta < x$ for all $x \in X$ and if x_1, x_2

ϵ X we use the order of X. By Theorem (2.9) there is a domain S such that Spec S $\cong$ Y. Let A be the product of all non-zero primes in S. Let R = S/A and we have Spec R $\cong$ X.

(2.11) <u>Discussion</u>. Let $f:X \to Y$ be an order preserving map between finite partially ordered sets. We can now construct rings R,S such that Spec R $\cong$ X and Spec S $\cong$ Y. It is natural to ask whether the rings R,S can be chosen in such a way that there exists a homomorphism $F:S \to R$ such that the diagram in Figure 10 commutes. (Spec F:Spec R$\to$Spec S is defined by Spec F(P) = $F^{-1}(P)$.)

$$\begin{tikzcd}
\text{Spec } R \arrow[r, "\text{Spec } F"] \arrow[d, "\cong"'] & \text{Spec } S \arrow[d, "\cong"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Figure 10

In [4], M. Hochster calls a topological space <u>spectral</u> if it is T_0; quasi-compact; the quasi-compact open sets are closed under finite intersection and form an open basis; and every non-empty irreducible closed subset has a generic point. Hochster notes that if R is a ring, Spec R is spectral. He then shows that, up to isomorphism, every spectral space is in the image of the functor Spec from the category of commutative rings with identity to the category of topological spaces and continuous maps. As we stated earlier, we can topologize a finite partially ordered set X by letting a subbasis for the closed sets be $\overline{\{x\}} = \{y \in X \mid y \geq x\}$. X will be a finite T_0 space and is thus trivially spectral. Thus Hochster's work applied to finite T_0 spaces gives a functorial version of our Theorem (2.10).

Hochster calls a continuous map spectral if the inverse image of a quasi-compact open set is a quasi-compact open set. If X,Y are spectral spaces and if $f: X \to Y$ is a spectral map, consider the category, A, consisting of X,Y as objects and morphisms f, 1_X, and 1_Y. Hochster shows that there exists a functor from A to the category of commutative rings which when followed by Spec gives a functor which is isomorphic to the inclusion functor from A to the category of spectral spaces and maps. This result gives an affirmative answer to our question in (2.11). Our technique for constructing rings does not give the necessary homomorphisms for this functorial result. We can illustrate the problem with the following example.

(2.12) <u>Example</u>. Let X be the partially ordered set in Figure 11 and Y the set in Figure 12. Let $f: X \to Y$ be defined by $f(\theta_X) = \theta_Y$ and $f(m_i) = p_i$.

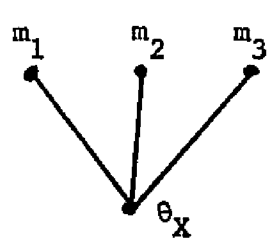


Figure 11

$X \xrightarrow{f} Y$

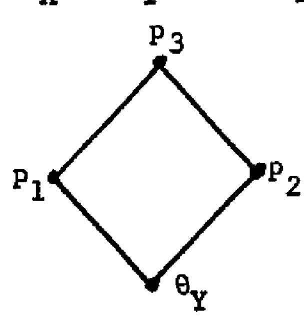


Figure 12

Let us assume that we have domains R,S such that Spec R = X, Spec S = Y and $F:S \to R$ is a ring homomorphism chosen such that Spec (F) = f. Since $f(\theta_X) = \theta_Y$ we must have ker F = 0 and thus F is an injection. Therefore, we may assume $S \subset R$ and that the map f merely contracts primes of R to S, i.e. $m_i \cap S = p_i$. There exists a rank 1 valuation ring $W \supset R$ such that the maximal ideal of W is centered on m_3; hence on p_3 in S. We can intersect W with the quotient field of S to obtain a valuation ring centered on p_3 and contained in the quotient field of S.

We will argue that our construction of S makes it impossible for a rank 1 valuation ring to be centered on the maximal ideal of S. Following our proof of Theorem (2.9), we choose independent rank 1 valuation rings V_1 and V_2 such that there are fields $k \subset K \subset V_1 \cap V_2$ and there is a rank 1 valuation ring V_3 of K over k. Let P_1 and P_2 be the maximal ideals of V_1 and V_2. Then we let $D = (V_3 + P_1) \cap (V_3 + P_2)$ and let $S = k + J(D)$. We get Spec S = Y. Let us assume that W is a rank 1 valuation ring of the quotient field of S with maximal ideal M centered on J(D). Since $P_1 \cap P_2 \subset J(D) \subset M$, V_1, V_2 and W are not independent. But if $V_1 = W$ then $J(D) \subset P_1$ and this is impossible. We conclude that there does not exist a rank 1 valuation ring centered on J(D) in S.

The use of the valuation rings V_1 and V_2 and the construction $V_3 + P_i$ has made it impossible for a rank 1 valuation ring to be centered on the maximal ideal of S. We do not know if it is possible to choose domains V_1 and V_2 delicately enough for our construction to produce the desired domains $S \subset R$ such that the diagram in Figure 10 will commute. However, if we could construct S in such a way that there is a rank 1 valuation ring W centered on p_3 then we would be able to construct the desired domain $R \supset S$. For there exist rank 1 valuation rings V_1 and V_2 centered on p_1 and p_2 in S and we can let $R = V_1 \cap V_2 \cap W$. We would then have Spec $R = X$ and $m_i \cap S = p_i$.

Chapter III

Prüfer and Bezout Domains

We will say that X is a <u>tree</u> if X is a partially ordered set with the following property. (*) If $x, y, z \in X$, $x \leq z$, and $y \leq z$ then either $x \leq y$ or $y \leq x$. Property (*) is equivalent to the statement that if $x \in X$, then $\{y \in X | y \leq x\}$ is a chain (i.e. a totally ordered set). We shall be interested in trees which satisfy Kaplansky's two properties for a partially ordered set X.

(K1) Every chain in X has a supremum (sup) and an infimum (inf).

(K2) If $x,y \in X$ and $x < y$ then there exist elements $x_1, y_1 \in X$ such that $x \leq x_1 < y_1 \leq y$ and there does not exist an element of X properly between x_1 and y_1.

We shall say that x_1 and y_1 are "immediate neighbors" and use the notation $x_1 \ll y_1$.

(3.1) <u>Theorem.</u> Let X be a partially ordered set. Then the following are equivalent.

(a) X is a tree with properties (K1), (K2) and a unique minimal element.

(b) There exists a Bezout domain D such that Spec $D \cong X$ (as a partially ordered set).

(c) There exists a Prüfer domain D such that Spec $D \cong X$ (as a partially ordered set).

Proof. (b) $\Rightarrow$ (c): Every Bezout domain is Prüfer.

(c) $\Rightarrow$ (a): D is a domain so we have (K1), (K2), and a unique minimal element for Spec D and thus for X. Since D is Prüfer, D_P is a valuation ring for any prime ideal $P \subset D$. The prime ideals of D_P form a chain so the same is true for the primes of D contained in P.

(a) $\Rightarrow$ (b): The remainder of this chapter develops the machinery necessary for this part of the proof.

We shall need a few definitions, most of which can be found in [5] or [9]. By an ordered group we will mean a pair (G, H) where G is an (additive) abelian group and H is a subset of G such that (i) $H + H \subset H$ and (ii) $H \cap (-H) = \{0\}$. We call H the positive elements of G and define an inequality relation by $f \leq g$ if and only if $g - f \in H$. We can easily see that if (G,H) is an ordered group, our relation is reflexive, anti-symmetric, transitive, and also that addition preserves the inequality. In practice we shall simply say that G is an ordered group and we will use G^+ to denote the positive elements of G.

Let g_1, g_2 be elements of the ordered group G. Then $g = \inf\{g_1, g_2\}$ means $g \leq g_1, g_2$ and if $h \leq g_1, g_2$ then $h \leq g$. We will use the notation $g = g_1 \wedge g_2$. If any pair of elements in G has an infimum, we say that G is a lattice group. If $G^+ \cup (-G^+) = G$, we say that G is totally ordered. If $g, h \in G$, an ordered group, then g and h are called disjoint if $g \wedge h = 0$. When working with an ordered group G which is not necessarily lattice, we use a replacement for the notion of infimum. We say $g_o \geq \inf_G\{g_1, \cdots, g_n\}$ if $g_o \geq g$ for all $g \in G$ such that $g \leq g_1, \cdots, g_n$.

Let Q be a subset of the ordered group G. We say that Q is a <u>segment</u> of G if $Q + G^+ \subset Q$ and Q is bounded below. A segment Q is <u>integral</u> if $Q \subset G^+$. An integral segment Q is a <u>prime segment</u> if $G^+ \setminus Q$ is closed under addition and $Q \neq G^+$. The segment Q is a <u>V-segment</u> if $g \geq \inf_G\{g_1, \ldots, g_n\}$ for any $g_1, \ldots, g_n$ in Q implies that $g \in Q$. If G is lattice, this condition simplifies to $g \wedge h \in Q$ whenever $g, h \in Q$.

Let D be a domain, and let K be its quotient field. We let D^* and K^* be the non-zero elements of each, and we let u(D) be the units of D. Let $A = K^*/u(D)$ (written additively). If $w: K^* \to A$ is the canonical map, we set $A^+ = w(D^*)$ and we get an ordered group. We call this ordered group the <u>group of divisibility</u> of D. One can easily see that w has the following properties:

(i) $w(xy) = w(x) + w(y)$.

(ii) $w(x+y) \geq \inf\{w(x), w(y)\}$, if $x + y \neq 0$.

(iii) $w(-1) = 0$.

Perhaps the best theorem involving groups of divisibility is the one that follows.

<u>Theorem</u> (Jaffard). Every lattice ordered group is a group of divisibility. We shall soon need this theorem. For its proof we refer the reader to [5, page 78, Theorem 3] or [9, page 586]. In [9], Ohm goes a step farther and shows that the domains obtained by Jaffard's construction are Bezout. This can also be found in [1, page 612].

It is an immediate result of the definition of a Bezout domain that the group of divisibility of a Bezout domain is always lattice. So let us assume that B is a Bezout domain, G its group of divisibility, and $w:B\to G$ the canonical map. Let $P \neq \{0\}$ be a prime ideal in B. By the first property of a group of divisibility, w(P) will be a prime segment. Now if $w(x_1)$, $w(x_2)$ are in w(P), choose $y \in B$ such that $(y) = (x_1,x_2)$ and we get $w(y) = w(x_1) \wedge w(x_2)$. Since y must be in P, $w(y) \in w(P)$ and w(P) is a prime V-segment. On the other hand if Q is a non-empty prime V-segment of G we see that $Q + G^+ \subset Q$ implies $B \cdot w^{-1}(Q) \subset w^{-1}(Q)$. Further, Q a V-segment and $w(x-y) \geq w(x) \wedge w(y)$ implies $w^{-1}(Q)$ is an ideal. That Q is a prime segment implies that $w^{-1}(Q)$ is a prime ideal. Thus there exists a one-to-one order preserving correspondance between the prime ideals of B and the prime V-segments of G. (The zero prime of B is associated with the empty set which is vaceously a prime V-segment of G.) We have proved the following theorem.

(3.2) Theorem. If B is a Bezout domain, and G its group of divisibility, then Spec B is isomorphic as a partially ordered set to the set of prime V-segments of G.

In view of the preceding discussion and theorem, we can complete Theorem (3.1) by constructing a lattice group such that the set of its prime V-segments is isomorphic to a prescribed tree with a unique minimal element for which (K1) and (K2) hold.

Construction of the Group

Let P be a partially ordered set, and let G be a non-trivial totally ordered group. Let $A = \{f:P\to G \mid f(p) = 0$ for all but a finite number of $p \in P\}$. If we define addition pointwise, A is an abelian group. Let $f \in A$. $\{p \in P \mid f(p) \neq 0\}$ is called the <u>support of f</u> and will be denoted $S(f)$. $\{p \in P \mid f(p) \neq 0$ and $f(s) = 0$ for all $s < p\}$ will be called the <u>minimal support of f</u> and denoted $MS(f)$.

(3.3) <u>Lemma</u>. Let A be the group defined above and let $A^+ = \{f \in A \mid f(p) > 0$ for all $p \in MS(f)\}$. Then A is an ordered group.

<u>Proof</u>. Since $MS(0) = \emptyset$, 0 satisfies our condition vaceously and $0 \in A^+$. Let $g,h \in A^+$. Choose $p \in MS(g+h)$. If there exists an $r < p$ for which $h(r) \neq 0$, we may assume that r is chosen in $MS(h)$ and thus $h(r) > 0$. But $r < p$ implies $(g+h)(r) = 0$ and so $g(r) < 0$. Choose $s \leq r$ such that $s \in MS(g)$. Then $g(s) > 0$ and $s < r$ so $h(s) = 0$ and we get $(g+h)(s) = g(s) > 0$. But $s < r < p$ and $p \in MS(g+h)$ so we have a contradiction. Thus either $h(p) = 0$ or $p \in MS(h)$ and $h(p) > 0$. The same reasoning allows us to conclude $g(p) = 0$ or $p \in MS(g)$ and $g(p) > 0$. Therefore $(g+h)(p) = g(p) + h(p) \geq 0$. Since $p \in MS(g+h)$, $(g+h)(p) > 0$.

Suppose now that $f \geq 0$ and $-f \geq 0$. We must have $MS(f) = MS(-f)$ so if $p \in MS(f)$ then $f(p)$ and $-f(p)$ are both greater than 0. Since G is totally ordered this cannot happen, so we conclude that $MS(f) = \emptyset$; hence $f = 0$. Thus A is an ordered group.

We will often need to compare two elements in A. By definition $f \geq g$ if and only if $f-g \geq 0$. From the way our order is defined in Lemma (3.2), this is equivalent to the following statement. If $f,g \in A$ then $f \geq g$ if and only if $f(s) = g(s)$ for all $s < p$ implies $f(p) \geq g(p)$.

If $f \in A^+$ and $MS(f)$ consists of exactly one element then we say that f is <u>irreducible</u>.

(3.4) <u>Lemma</u>. If P is a tree and A is the ordered group of Lemma (3.3) then A is a lattice group. If, in addition, $f > 0$ then there exists a unique set $\{f_1, \ldots, f_n\}$ of pairwise disjoint irreducible elements of A such that $f = f_1 + \cdots + f_n$.

<u>Proof</u>. Let $f, g \in A$ and let $H = \{s \in P \mid f(s) \neq g(s)$ and $f(r) = g(r)$ if $r < s\}$. Since P is a tree, if $r \in P$ then there is at most one $s \in H$ such that $s \leq r$. We define an element $h \in A$ as follows:

(i) $h(p) = f(p) = g(p)$ if there does not exist an $s \in H$ such that $s \leq p$.

(ii) $h(p) = f(p)$ if $s \in H$, $s \leq p$ and $f(s) < g(s)$.

(iii) $h(p) = g(p)$ if $s \in H$, $s \leq p$ and $g(s) < f(s)$.

Having defined h in this way we can observe that for any $p \in P$, either $h(s) = f(s)$ for all $s \leq p$ or $h(s) = g(s)$ for all $s \leq p$.

<u>Claim</u>. $h = f \wedge g$.

To see that $f \geq h$ let $p \in P$ and suppose $f(s) = h(s)$ for all $s < p$. If $h(p) = f(p)$, fine. If not, we must have $h(s) = g(s)$ for all $s \leq p$. But this also means $f(s) = g(s)$ for all $s < p$ and $f(p) \neq g(p)$. By the way h was defined, $h(p) = g(p) < f(p)$. Thus $f \geq h$. Likewise $g \geq h$, so suppose $h' \leq f$ and $h' \leq g$. We wish to show $h' \leq h$. Suppose $p \in P$ and for all $s < p$, $h'(s) = h(s)$. Either $h(s) = f(s)$ for all $s \leq p$ or $h(s) = g(s)$ for all $s \le p$. Either case implies $h'(p) \le h(p)$. Thus $h' \le h$ and $h = f \wedge g$. Thus A is lattice.

Now let $f > 0$ and let $MS(f) = \{p_1, \cdots, p_n\}$. We define f_i, the p_i^{th} associated component of f as follows:

$$\text{for } p \in P,\ f_i(p) = \begin{cases} f(p) & \text{if } p \ge p_i \\ 0 & \text{if } p \not\ge p_i. \end{cases}$$

For each i, $MS(f_i) = \{p_i\}$; thus f_i is irreducible. For $p \in P$, if $f(p) = 0$ then $f_i(p) = 0$ for all i. Since P is a tree, if $p \in P$ and $f(p) \neq 0$ then there is a unique $p_i \le p$. Thus $f_i(p) = f(p)$ and if $j \neq i$ then $p_j \not\le p$ and thus $f_j(p) = 0$. Thus $f = \sum_{i=1}^{n} f_i$.

Choose $i \neq j$ and define $H_{ij} = \{s \in P \mid f_i(s) \neq f_j(s)$ and $f_i(r) = f_j(r)$ if $r < s\}$. Then $H_{ij} = \{p_i, p_j\}$. Thus if $p \ge p_i$ then $(f_i \wedge f_j)(p) = f_j(p) = 0$, if $p \ge p_j$ then $(f_i \wedge f_j)(p) = f_i(p) = 0$, and if $p \not\ge p_i$ and $p \not\ge p_j$ then $(f_i \wedge f_j)(p) = f_i(p) = f_j(p) = 0$. We conclude $f_i \wedge f_j = 0$.

Suppose $g_1, \cdots, g_m$ are pairwise disjoint irreducible elements of A and that $\sum_{i=1}^{m} g_i = f$. Let $r_i = MS(g_i)$. Since the g_i's are disjoint, $\{r_1, \cdots, r_m\} = MS(f) = \{p_1, \cdots, p_n\}$. Thus $n = m$ and we may assume $r_i = p_i$. Let $p \in P$ and $p \ge p_j$ where $p_j \in MS(f)$. Then $f_j(p) = \sum_{i=1}^{n} f_i(p) = \sum_{i=1}^{n} g_i(p) = g_j(p)$. If $p \not\ge p_j$, $f_j(p) = 0 = g_j(p)$. Thus $f_j = g_j$ for each i, and our decomposition is unique.

(3.5) <u>Theorem</u>. Let P be a tree with a unique minimal element such that (K1) and (K2) hold. Then there exists a lattice group A such that the set of prime V-segments, $Q(A)$, ordered by inclusion is isomorphic to P.

Proof. We will let θ be the unique minimal element of P. Let $P^* = \{p \in P \mid$ there exists a $q \ll p\}$. We give P^* the induced order of P and thus P^* is a tree. The following fact will soon prove useful.

If $p \neq \theta \in P$, then $p = \sup_P\{q \in P^* \mid q \leq p\}$. To see this we use (K1),(K2) and the fact that P is a tree. Since P is a tree, $\{q \in P^* \mid q \leq p\}$ is a chain and (K1) tells us $\sup_P\{q \in P^* \mid q \leq p\}$ exists. Of course it must be $\leq p$. Let $q_1 < p$. Then using (K2) we can choose $r, s \in P$ such that $q_1 \leq r \ll s \leq p$. Thus $s \in P^*$ and $q_1 < s \leq p$ so $q_1 \neq \sup_P\{q \in P^* \mid q \leq p\}$. Thus p must work.

Now let $A = \{f: P^* \to Z \mid f(p) = 0$ for all but a finite number of $p \in P^*\}$ where Z is the integers with the usual ordering. As in Lemma (3.3) we define $A^+ = \{f \in A \mid f(p) > 0$ for all $p \in MS(f)\}$. Since P^* is a tree, we can apply Lemma (3.4) to conclude that A is lattice. As we said earlier, we consider the empty set, $\emptyset$, to be a prime V-segment of A. Thus it is contained in all the others and is the minimal element of $Q(A)$.

Now for any $p \in P$ we define $Q_p = \{f \in A^+ \mid$ there exists $s \in MS(f)$ such that $s \leq p\}$.

Claim. Q_p is a prime V-segment.

$\theta \notin P^*$ implies $Q_\theta = \emptyset$ so let us assume $p \neq \theta$. Since $\theta < p$, there is an $s \in P^*$ such that $\theta < s \leq p$. If we define $f_s(q) = \begin{cases} 1 & \text{if } q = s \\ 0 & \text{otherwise} \end{cases}$, then $f_s \in Q_p$ and $Q_p \neq \emptyset$. Let $f \in Q_p$ and $h \in A^+$. Then there is an $s \in MS(f)$ such that $s \leq p$. If $h(r) = 0$ for all $r < s$ then $s \in MS(f+h)$. Otherwise, let q be the minimal element of the chain $\{r \in P^* \mid r < s$ and $h(r) \neq 0\}$. Since $q < s$ and $s \in MS(f)$, we will have $q \in MS(f+h)$ and $q < p$. In either case we get

$f + h \in Q_p$. Thus Q_p is a segment.

Now if $f,h \in A^+\backslash Q_p$, then $f(s) = 0 = h(s)$ for all $s \leq p$. Therefore $(f+h)(s) = 0$ for all $s \leq p$. Hence $f + h \in A^+\backslash Q_p$ and Q_p is a prime segment.

Suppose $f,g \in Q_p$. We may assume without loss of generality that for all $s \leq p$, $(f\wedge g)(s) = f(s)$. Choose $q < p$ such that $q \in MS(f)$. For all $s < q$, $(f\wedge g)(s) = f(s) = 0$ and $(f\wedge g)(q) = f(q) > 0$. Thus $q \in MS(f\wedge g)$ and $f \wedge g \in Q_p$. Thus we have shown that Q_p is a prime V-segment.

We now define a map $\Phi : P \to Q(A)$ by $\Phi(p) = Q_p$. We must show that Φ is an isomorphism of partially ordered sets. It is immediate from the definition that Φ is order preserving, so let $p,q \in P$ and assume $\theta < p \nleq q$. Since $p = \sup_P\{s \in P^* | s \leq p\}$, there is an $s \leq p$ such that $s \in P^*$ and $s \nleq q$. We define an element 1_s by $1_s(r) = \begin{cases} 1 \text{ if } r = s \\ 0 \text{ otherwise} \end{cases}$. Then $1_s \in Q_p$ and $1_s \notin Q_q$. Thus $Q_p \not\subset Q_q$; hence $Q_p \neq Q_q$ and Φ is injective. This also shows that if $Q_p \subset Q_q$ then $p \leq q$ so our proof will be complete when we show Φ is surjective.

We know $\Phi(\theta) = \emptyset$ so let Q be a non-empty prime V-segment of A. Let $f \in Q$, $MS(f) = \{p_1, \cdots, p_n\}$, and $f_1, \cdots, f_n$ the associated components of f. Since Q is a prime segment, at least one $f_i \in Q$. But $0 \notin Q$ and if $i \neq j$, $f_i \wedge f_j = 0$; so Q is a V-segment implies only one f_i can be in Q. Let $Y = \{p_j \in P^* |$ there is an $f \in Q$ for which $p_j \in MS(f)$ and the associated component $f_j \in Q\}$. Now suppose $p_i,q_j \in Y$ as a result of elements $f_i,h_j \in Q$ where $MS(f_i) = \{p_i\}$ and $MS(h_j) = \{q_j\}$. Then $f_i \wedge h_j \in Q$ since Q is a V-segment. If $p \in P^*$ and $p_i \nleq p$ then $f_i(s) = 0$ for all $s \leq p$, hence $(f_i\wedge h_j)(p) = 0$. If $p \in P^*$ and $q_j \nleq p$ then $h_j(s) = 0$ for all $s \leq p$, hence $(f_i\wedge h_j)(p) = 0$. Thus if $(f_i\wedge h_j)(p) \neq 0$ then $p_i \leq p$ and $q_j \leq p$. Since $f_i \wedge h_j \neq 0$ there must exist at least one $p \in P^*$ such that $(f_i\wedge h_j)(p) \neq 0$;

hence either $p_i \leq q_j$ or $q_j \leq p_i$. Thus Y is a chain. Let $y = \sup_P Y$. (Property (K1) tells us that the sup exists.)

Claim. $Q = Q_y$.

If $f \in Q$ then there is a $p_i \in MS(f)$ such that $p_i \in Y$; thus $p_i \leq y$ and $f \in Q_y$. Therefore $Q \subset Q_y$. Let $f \in Q_y$. Since Q_y is a prime V-segment, there is exactly one associated component, f_i, of f which is in Q_y. Of course $f \geq f_i$. Now if $MS(f_i) = \{p_i\}$, then $p_i \leq y$ and $p_i \in P^*$. Thus there is a $q \ll p_i$ and $q \neq \sup_P Y = y$. Thus there exists $s \in Y$ such that $q < s \leq y$. Now both p_i and s are $\leq y$ and $s \nless p_i$ since $s < p_i$ would lead to $q < s < p_i$ and we know $q \ll p_i$. We have then $p_i \leq s \leq y$. Since $s \in Y$, there is a $g \in Q$ such that $MS(g) = \{s\}$. Now $f_i(p_i) > 0$ so there is a positive integer n such that $n(f_i(p_i)) > g(s)$. Therefore for all $r < p_i$, $n(f_i(r)) = 0 = g(r)$ and $n(f_i(p_i)) > g(p_i) = \begin{cases} 0 \text{ if } p_i < s \\ g(s) \text{ if } p_i = s \end{cases}$. We conclude $nf_i \geq g$. Since Q is a segment and $g \in Q$, we see that $nf_i \in Q$. Since Q is prime we have $f_i \in Q$ and thus $f \in Q$. Thus $Q_y \subset Q$; hence $Q = Q_y$ and Φ is surjective. This completes our proof.

We can now prove (a) $\Rightarrow$ (b) in Theorem (3.1).

(3.1) Theorem (a) $\Rightarrow$ (b). If X is a tree with a unique minimal element such that (K1) and (K2) hold, then there exists a Bezout domain D such that Spec $D \cong X$.

Proof. We combine Theorem (3.5), Jaffard's Theorem, and Theorem (3.2) to obtain the necessary Bezout domain.

A slight generalization of (3.1) exists.

(3.6) Corollary. Let X be a partially ordered set. Then X is a tree for which (K1) and (K2) hold and X has exactly n minimal elements if and only if there is a ring R such that R is the direct sum of n non-trivial Bezout (Prüfer) domains and Spec $R \cong X$.

Proof. The proof depends upon two facts. First, if X has exactly n minimal elements then X is the disjoint union of n trees for which (K1) and (K2) hold and which have a unique minimal element. Second, if a ring R is the direct sum of rings $R_1, \ldots, R_n$ then Spec R is isomorphic to the disjoint union of Spec $R_1, \ldots,$ Spec R_n. Let us emphasize that for a partially ordered set X to be the disjoint union of X_1 and X_2, no element of X_1 can be related to an element of X_2 by the order on X. The first fact is true because X is a tree and so no element can be bigger than two different minimal elements. The second fact comes from [11, page 75, Theorem 30]. We see there that if R is the direct sum of $R_1, \ldots, R_n$ then any ideal A of R can be written as $A_1 + \cdots + A_n$ where either $A_i = R_i$ or A_i is an ideal of R_i and A is prime if and only if all but one of the A_i coincide with the corresponding R_i and the remaining A_i is prime in R_i.

Chapter IV

Infimum and Supremum Properties

If X is a partially ordered set and S a subset, S is lower directed if whenever $a \in S$ and $b \in S$, then there must be a $c \in S$ such that $c \le a$ and $c \le b$. Upper directed sets are defined similiarly.

(4.1) Theorem. Let $\{P_\alpha\}$ be a set of prime ideals in a ring R. If $\{P_\alpha\}$ is lower directed, $\cap P_\alpha$ is a prime ideal and if $\{P_\alpha\}$ is upper directed, $\cup P_\alpha$ is a prime ideal.

Proof. Suppose $\{P_\alpha\}$ is lower directed. If $a,b \notin \cap P_\alpha$ then there are prime ideals P,Q such that $a \notin P$, $b \notin Q$. Since there must be a prime ideal $P' \subset P \cap Q$, $a,b \notin P'$ and so $ab \notin P'$. Thus $ab \notin \cap P_\alpha$.

If $\{P_\alpha\}$ is upper directed and $a,b \in \cup P_\alpha$ then $a-b \in \cup P_\alpha$ since there must be a P_α such that $a,b \in P_\alpha$. Obviously if $r,s \notin \cup P_\alpha$ then $rs \notin \cup P_\alpha$.

Thus for any ring R, Spec R has the property:

(D) Every lower directed set has a greatest lower bound and every upper directed set has a least upper bound.

In [4, Proposition 5], Hochster also observes that the spectrum of a ring has this property. Thus all the properties (D), (K1) and (K2) are valid for the partially ordered set Spec R. It is obvious that (D) $\Rightarrow$ (K1), but we do not know if (K1) $\Rightarrow$ (D) or even if (K1) and (K2) $\Rightarrow$ (D). We will show that (K1) $\Rightarrow$ (D) for any partially ordered set which is either countable or lattice. Thus an example which shows (K1) $\not\Rightarrow$ (D) (if one exists) would have to be an uncountable non-lattice set.

(4.2) Theorem. If X is a countable partially ordered set with property (K1) then (D) holds in X.

Proof: Suppose $\{x_n\}$ is a lower directed set in X. We define a chain as follows:

$y_1 = x_1$

y_2 = the first element of $\{x_n\}$ which is less than y_1 and x_2.

(y_2 exists because $\{x_n\}$ is lower directed.)

y_i = the first element of $\{x_n\}$ which is less than y_{i-1} and x_i.

Clearly $\{y_i\}$ is a chain. By (K1) there is a $y \in X$ such that $y = \inf\{y_i\}$. Clearly $y \le x_n$ for all x_n and if $z \le x_n$ for all x_n then $z \le y_i$ for all i and so $z \le y$. Thus y is the greatest lower bound of $\{x_i\}$. The proof for upper directed sets is similiar.

We recall that a partially ordered set X is lattice if the inf and sup of any two elements exist and that X is a complete lattice if every subset of X has a sup and an inf. We obtain the following characterization of a complete lattice.

(4.3) Theorem. Let X be a lattice set. Then the following are equivalent.

(a) X is a complete lattice.

(b) X has property (D).

(c) X has property (K1).

<u>Proof</u>. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) Trivial.

(c) $\Rightarrow$ (a). Let $P \subset X$ and let $L = \{\ell \in X | \ell \leq p \text{ for all } p \in P\}$.

Let $\overline{P} = \{x \in X | x \geq \ell \text{ for all } \ell \in L\}$. Thus $P \subset \overline{P}$ and $\ell \in L$ if and only if $\ell \leq x$ for all $x \in \overline{P}$. If $a,b \in \overline{P}$ then we must have $a \wedge b \in \overline{P}$. Let M be a maximal chain in $\overline{P}$ and let $z = \inf M$, since (K1) holds. If $\ell \leq x$ for all $x \in \overline{P}$ then $\ell \leq x$ for all $x \in M$ since $M \subset \overline{P}$. Thus $\ell \leq z$ and we can say $z \in \overline{P}$. If $x \in \overline{P}$ then $x \wedge z \in \overline{P}$ and $x \wedge z \leq z$. Since M is maximal in $\overline{P}$ we cannot have $x \wedge z < z$. We conclude that $z = x \wedge z$ and thus $z \leq x$. Thus $z = \inf \overline{P}$. Clearly $z \leq p$ for $p \in P$ and if $\ell \in L$ we know $\ell \leq z$. Thus $z = \inf P$. Similiarly we can show that the sup of any subset of X exists so we conclude X is a complete lattice.

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